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Giant Magnons on $\mathbb{CP}^3$ by Dressing Method

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Abstract

We consider classical string spectrum of $\mathbb{R}_t \times \mathbb{CP}^3$, and construct a family of solutions with residual $SU(2)$ symmetry by the dressing method on $SU(4)/U(3)$ sigma model. All of them obey the square-root type dispersion relation often found in the theory with $su(2|2)$ symmetry. A single dyonic giant magnon is not found in this approach.

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1 Introduction

An effective field theory of coincident membranes with $\mathcal{N} = 8$ superconformal symmetry in 1+2 dimensions is proposed by Bagger, Lambert, and Gustavsson (BLG) based on three-algebra [1, 2, 3]. Aharony, Bergman, Jafferis and Maldacena (ABJM) proposed an $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory with a tunable coupling constant $\lambda = N/k$ by generalizing the BLG theory to incorporate $U(N)_k \times U(N)_{-k}$ symmetry group — which coincides with the special case of [4, 5] — and argued that their model at the 't Hooft limit is dual to type IIA superstring theory on the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background [6, 7].

The IIA on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is less supersymmetric than the IIB on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which was conjectured to be dual to $\mathcal{N} = 4$ super Yang-Mills in 1+3 dimensions [8]. Integrability has provided us a powerful tool to study the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, and a matter of central concern is whether and how similar techniques are applied to the $\mathrm{AdS}_4/\mathrm{CFT}_3$ case.

Despite huge and rapid progress on this subject, no conclusive answer has been given. Looking on the positive side, one finds the integrability of two-loop Hamiltonian in ABJM model [9, 10, 11, 12], classical integrability of superstring action (except for a subtle issue concerning strings in AdS_4) [13, 14, 15],[1] and the proposal of all-loop Bethe Ansatz [20, 21, 22, 23], which is consistent with near-plane wave limit of string theory [24, 25, 26, 27, 28]. On the negative side, one finds disagreement between the one-loop energy of folded or circular string, and the proposed Bethe Ansatz [29, 30, 31, 32, 33].

More data, especially the examples that are not found in $\mathrm{AdS}_5 \times \mathrm{S}^5$ case, are necessary to refine our understanding of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality and its integrability [34, 35, 36, 37, 38, 39, 40, 41, 42]. A good starting point will be to reconsider the correspondence between magnons in the asymptotic spin chain and giant magnon solutions on the decompactified worldsheet [43, 44, 45, 25], as one could expect nice examples of the duality owing to the $su(2|2)$ symmetry.

The description by algebraic curve tells the classical string spectrum and its dispersion in a simple way [46, 47]. Yet, to obtain further information such as (semiclassical) quantization and scattering [48, 49, 50, 51, 52],[2] it is useful to construct an explicit profile of the corresponding string solution.

The aim of this paper is to construct the explicit profile of classical strings on which only the existence and the dispersion have been known so far by means of algebraic curve. The relationship between a string solution and an algebraic curve is not explicit in general, so we have to construct the classical string solution from scratch. This sort of problem is quite difficult in general, due to the nonlinearity of differential equations. Here the integrability helps. In particular, the dressing method enables us to construct soliton solutions of integrable

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[1]The pure spinor action for $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is constructed in [16, 17, 18], which remains superconformal at quantum level as in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ case [19].

[2]There are proposals to perform quantization in the language of algebraic curve [53, 54].

classical sigma models by means of linear algebra. The dressing method was developed in [55, 56, 57, 58, 59]. Application to Grassmannian sigma models including $\mathbb{CP}^N$ cases was intensively studied in [60, 61, 62]. This method has also been applied to $\mathbb{R}_t \times S^5$ sigma model successfully in [63, 64] to construct an explicit profile of multi giant magnon solutions.

Unfortunately, it is not guaranteed that the dressing method exhausts all solutions. In the dressing method, one chooses the vacuum solution and a particular embedding of $\mathbb{CP}^N$ into $SU(N)$ principal chiral model for some N. As is found by [63, 64], there are some solutions which can be obtained in one embedding, but cannot in another embedding.

In our case, we can obtain solutions with residual $SU(2)$ symmetry which carries only two nonvanishing components of angular momenta. However, we do not find a single "small" dyonic solution [47], the one living on the $\mathbb{CP}^2$ subspace, and smoothly connected to $\mathbb{CP}^1$ giant magnons. We need to refine the dressing method to construct $\mathbb{CP}^2$ dyonic giant magnons.

The paper is organized as follows. We explain some preliminaries in Section 2. In Section 3 we construct the solutions by dressing $SU(4)/U(3)$ model and discuss the limitations of our approach. Section 4 is for summary and discussions. In Appendix A, we discuss how to relate the string motion with some sine-Gordon fields by means of Pohlmeyer-Lund-Regge reduction. We briefly comment on the dressing of $SO(6)/U(3)$ model in Appendix B.

Note added: While the paper is in preparation, we find the paper [65, 66] on arXiv, which has an overlap with our result. Their results seem to be equivalent to (3.32).

2   Sigma model on $\mathbb{CP}^3$

2.1   Action in conformal gauge

We consider a classical σ-model on $\mathbb{R}_t \times \mathbb{CP}^3$, particularly the one obtained by large k limit of $S^7/\mathbb{Z}_k$. When the $\mathbb{Z}_k$ action becomes gauging of S^1, we obtain the $\mathbb{CP}^3$ space. We set the radius of S^7 to unity, and introduce an embedding $S^7 \subset \mathbb{C}^4$. The coordinates on $\mathbb{C}^4$ can be identified as the homogeneous coordinates of the $\mathbb{CP}^3$, normalized as

$$\sum_{i=1}^{4} \bar{z}_i \, z_i = 1. \tag{2.1}$$

Since the $\mathbb{CP}^3$ space has local scale invariance, we must identify two points z_i and cz_i with $c \in \mathbb{C}$. The condition (2.1) partially fixes the gauge, and the residual symmetry is $U(1)$.

The σ-model action in conformal gauge compatible with (2.1) is given by

$$S = -2h \int d^2\sigma \left[\gamma^{\alpha\beta} \left\{ -\frac{1}{4} \left(\partial_\alpha t \right) \left(\partial_\beta t \right) + (D_\alpha z_i)^\dagger D_\beta z_i \right\} + \Lambda \left(|z_i|^2 - 1 \right) \right], \tag{2.2}$$

where h is the string tension,[3] Λ a Lagrange multiplier. The worldsheet metric is $\gamma^{\tau\tau} = -\gamma^{\sigma\sigma} = -1$, $\gamma^{\sigma\tau} = \gamma^{\tau\sigma} = 0$. Covariant derivatives acts on fields as

$$D_\alpha X = \partial_\alpha X - i A_\alpha X, \qquad \overline{D_\alpha} X = \partial_\alpha X + i A_\alpha X. \tag{2.3}$$

Path integral of A_α and Λ gives,

$$A_\alpha = \frac{i\left(z_i\,\partial_\alpha \bar{z}_i - \bar{z}_i\,\partial_\alpha z_i\right)}{2\bar{z}_i z_i} = -i\bar{z}_i\,\partial_\alpha z_i\,. \tag{2.4}$$

The equations of motion are[4]

$$0 = D_\alpha^2 z_i + |D_\alpha z_j|^2\, z_i\,, \tag{2.6}$$

and Virasoro constraints,

$$\frac{\kappa^2}{4} = |D_\sigma z_i|^2 + |D_\tau z_i|^2 \;\; = |\partial_\sigma z_i|^2 + |\partial_\tau z_i|^2 - \left(A_\sigma^2 + A_\tau^2\right), \tag{2.7}$$

$$0 = \mathrm{Re}\left\{(D_\sigma z_i)^\dagger\, D_\tau z_i\right\} = \mathrm{Re}\left\{\partial_\sigma \bar{z}_i\,\partial_\tau z_i\right\} - A_\sigma A_\tau\,, \tag{2.8}$$

for $t = \kappa\tau$. Any solution of the equations of motion and Virasoro constraints on $\mathbb{CP}^m$ ($m < 3$) subspace can be lifted to a solution of $\mathbb{CP}^3$ [67].

In general, the $U(1)$ field strength is not zero:

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha = -i\left(\partial_\beta \bar{z}_i\,\partial_\alpha z_i - \partial_\beta z_i\,\partial_\alpha \bar{z}_i\right) \neq 0, \tag{2.9}$$

and therefore one cannot achieve $A_\tau = A_\sigma = 0$ by any gauge transformation.

The conserved charges are defined by

$$E = h \int d\sigma\,(\partial_\tau t)\,, \qquad J_{\ell m} = 4h \int d\sigma\,\mathrm{Im}\left(\bar{z}_\ell\, D_\tau z_m\right). \tag{2.10}$$

They satisfy $\sum_k J_{kk} = 0$. We use the notation $J_k = J_{kk}$ when the off-diagonal components of $J_{\ell m}$ are zero. This is indeed the case regarding all solutions discussed below.

2.2 Coset embedding

The $\mathbb{CP}^3$ sigma model can be embedded into $SU(N)$ principal chiral models in several ways. We have to choose one embedding to apply the dressing method. The simplest is $\mathbb{CP}^3 \subset SU(4)$. Let θ be an involution of $SL(4)$ given by $\theta = \mathrm{diag}\,(1, -1, -1, -1)$, then $\mathbb{CP}^3$ is characterized by

$$\mathbb{CP}^3 = \frac{SU(4)}{S\left[U(3) \times U(1)\right]} = \left\{g \in SU(4)\,|\,\theta g \theta g = \mathbf{1}_4\right\}, \tag{2.11}$$

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[3]$h(\lambda) = \sqrt{\lambda/2} + \cdots$ for $\lambda \gg 1$.

[4]If one expands the covariant the derivative, one finds

$$D_\alpha^2 z_i = \partial_\alpha^2 z_i - 2i A^\alpha \partial_\alpha z_i - i(\partial_\alpha A^\alpha) z_i - A_\alpha^2 z_i\,, \qquad |D_\alpha z_j|^2 = |\partial_\alpha z_j|^2 - A_\alpha^2\,. \tag{2.5}$$

where $\mathbf{1}_N$ is an $N \times N$ identity matrix. Equivalently, elements of $\mathbb{CP}^3$ can be written as

$$g \in \mathbb{CP}^3 \subset SU(4) \quad \Longleftrightarrow \quad g = \theta \left(\mathbf{1}_4 - 2P\right), \quad \text{with} \quad P = P^2, \ P^\dagger = P. \tag{2.12}$$

Suppose that the projector P has rank one whose image is spanned by $z = (z_1, \ldots, z_{N+1})^T$. We identify this vector z with the homogeneous coordinates of $\mathbb{CP}^N$ normalized to $z^\dagger z = 1$. The projector is written as

$$P_{ij} = z_i \, \bar{z}_j \,, \tag{2.13}$$

which is manifestly gauge-invariant. From this explicit relation, one can rewrite the $\mathbb{R}_t \times \mathbb{CP}^N$ action as [68][5]

$$S = 2h \int d^2\sigma \left\{ -\frac{1}{4} \left(\partial_a t\right)^2 + \frac{1}{2} \operatorname{tr}\left[(\partial_a P)^2\right] + \Lambda \left(P^2 - P\right) \right\}. \tag{2.14}$$

The equation of motion is

$$\left[\partial_a^2 P, P\right] = 0, \tag{2.15}$$

and Virasoro constraints are

$$\operatorname{tr}\left[(\partial_\pm P)^2\right] = \frac{\kappa^2}{2}\,, \qquad \text{for} \quad t = \kappa\tau, \tag{2.16}$$

where we introduced light-cone coordinates by $\tau = \sigma^+ + \sigma^-$, $\sigma = \sigma^+ - \sigma^-$.

We may rewrite the equation of motion as $\partial^a \left[\partial_a P, P\right] = 0$; this is the conservation law for the currents

$$j_a \equiv g^{-1}\partial_a g \quad = -2\left[\partial_a P, P\right] \quad = 2\left[z\left(D_a z\right)^\dagger - \left(D_a z\right) z^\dagger\right], \tag{2.17}$$

$$\ell_a \equiv -\partial_a g \, g^{-1} = -2\theta\left[\partial_a P, P\right]\theta = 2\theta\left[z\left(D_a z\right)^\dagger - \left(D_a z\right) z^\dagger\right]\theta. \tag{2.18}$$

They define the conserved charges as,

$$J_{\ell m} \equiv ih \int d\sigma \, (j_\tau)_{\ell m} = -2ih \int d\sigma \left[\partial_\tau P, P\right]_{\ell m} = 4h \int d\sigma \, \operatorname{Im}\left(\bar{z}_\ell D_\tau z_m\right). \tag{2.19}$$

The $\mathbb{CP}^3$ model can also be embedded into the $SU(6)$ principal chiral model, through $SO(6) \subset SU(6)$ and

$$\mathbb{CP}^3 \simeq \frac{SO(6)}{U(3)} = \{g \in SO(6) \,|\, KgKg = -\mathbf{1}_6\}, \tag{2.20}$$

where K is an antisymmetric involution of $SL(6)$. An explicit description of (2.20) is given in [13]. They found

$$g \equiv e^Y = \mathbf{1} + \sin\rho \, \tilde{Y} + (1 - \cos\rho) \, \tilde{Y}^2, \qquad \tilde{Y} \equiv \frac{Y}{\rho}, \qquad \rho^2 = \sum_{i=1}^{6} y_i^2. \tag{2.21}$$

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[5]There is an identity $\left[(D_a z)^\dagger(D_b z) + (D_b z)^\dagger(D_a z)\right] = \operatorname{tr}\left[\partial_a P \, \partial_b P\right] = -\frac{1}{4}\operatorname{tr}\left[j_a j_b\right]$.

where

$$Y = \begin{pmatrix} 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ 0 & 0 & y_2 & -y_1 & y_4 & -y_3 \\ -y_1 & -y_2 & 0 & 0 & y_5 & y_6 \\ -y_2 & y_1 & 0 & 0 & y_6 & -y_5 \\ -y_3 & -y_4 & -y_5 & -y_6 & 0 & 0 \\ -y_4 & y_3 & -y_6 & y_5 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.22}$$

The coordinates $\tilde{y}_i \equiv y_i/\rho$ are related to the normalized homogeneous coordinates on $\mathbb{CP}^3$ as follows:

$$(z_1, z_2, z_3, z_4) = \left(\sin\rho \, (\tilde{y}_1 + i\tilde{y}_2), \sin\rho \, (\tilde{y}_3 + i\tilde{y}_4), \sin\rho \, (\tilde{y}_5 + i\tilde{y}_6), \cos\rho \right). \tag{2.23}$$

Finally, there is an embedding $\mathbb{RP}^3 \simeq SU(2) \times \overline{SU(2)} \subset SU(4)$, defined by

$$g = \begin{pmatrix} g_2 & 0 \\ 0 & \bar{g}_2 \end{pmatrix}, \qquad g_2 = \sqrt{2} \begin{pmatrix} z_1 & -iz_2 \\ -i\bar{z}_2 & \bar{z}_1 \end{pmatrix}, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_4 \\ \bar{z}_3 \end{pmatrix}, \qquad \sum_{j=1}^{4} |z_j|^2 = 1. \tag{2.24}$$

2.3 Examples of spectrum in the decompactification limit

Soliton-like solutions on the $\mathbb{CP}^3$ sigma model can be found in the decompactification limit. In conformal gauge, this limit can be achieved by rescaling the worldsheet coordinates by $(\tilde{\tau}, \tilde{\sigma}) = (\mu\tau, \mu\sigma)$ with $\mu \to \infty$. Below we will discuss examples of the classical strings which obey the boundary condition

$$\left(\frac{z_1}{z_4}, \frac{z_2}{z_4}, \frac{z_3}{z_4} \right) \to \left(e^{it+ip_\pm}, 0, 0 \right), \qquad p \equiv \frac{p_+ - p_-}{2}, \tag{2.25}$$

as $\tilde{\sigma} \to \pm\infty$. See [67] for a thorough discussion.

2.3.1 Pointlike strings

The simplest solution is the pointlike string, or the geodesic in $\mathbb{CP}^3$, given by

$$t = \omega\tau, \quad (z_1, z_2, z_3, z_4) = \left(\frac{1}{\sqrt{2}} e^{i\omega\tau/2}, 0, 0, \frac{1}{\sqrt{2}} e^{-i\omega\tau/2} \right), \tag{2.26}$$

whose conserved charges are

$$E = J_1 = -J_4 = h \, 2\pi\omega. \tag{2.27}$$

Note that the profile $t = \omega\tau$, $(z_1, z_2, z_3, z_4) = (e^{i\omega\tau}, 0, 0, 0)$ is a solution but meaningless, because the Virasoro constraints imposes $\omega = 0$.

From the viewpoint of AdS/CFT, the pointlike string should correspond to the BPS states of the ABJM model,

$$\mathcal{O} = \text{tr} \left[Y^1 (Y_4)^\dagger Y^1 (Y_4)^\dagger \dots \right]. \tag{2.28}$$

If J_1, and J_4 are the numbers of Y_1, Y_4, the conformal dimension of (2.28) satisfies

$$\Delta = \frac{J_1 - J_4}{2}, \tag{2.29}$$

in agreement with (2.27). We can consider magnon excitations over the BPS vacuum. In the case of $SU(2) \times SU(2)$ sector, they are given by the replacement

$$Y^1 \;\rightarrow\; Y^2 \text{ or } Y^3, \qquad (Y_4)^\dagger \;\rightarrow\; (Y_3)^\dagger \text{ or } (Y_2)^\dagger. \tag{2.30}$$

2.3.2   Recycling solutions of $\mathbb{R}_t \times \mathbf{S^3}$

Any classical string solution on S^3 can be mapped to a solution on $\mathbb{RP}^3$. Let (t_S, ξ_1, ξ_2) be a classical string solution on $\mathbb{R}_t \times S^3$ with $|\xi_1|^2 + |\xi_2|^2 = 1$. Then the ansatz

$$t = 2t_S, \qquad (z_1, z_2, z_3, z_4) = \frac{1}{\sqrt{2}} \left(\xi_1, \xi_2, \xi_2^*, \xi_1^* \right), \tag{2.31}$$

is a consistent solution of $\mathbb{R}_t \times \mathbb{RP}^3$. The boundary condition (2.25) is translated into

$$(\xi_1, \xi_2) \rightarrow \left(\exp\left[it_S + \frac{ip_\pm}{2} \right], 0 \right), \quad p \equiv \frac{p_+ - p_-}{2} \qquad \text{as } \tilde{\sigma} \rightarrow \pm\infty. \tag{2.32}$$

If ξ_2 is real, we can easily generalize the ansatz (2.31) by using the $SU(2)$ symmetry acting on (z_2, z_3) [37, 34].

From strings living on $\mathbb{R}_t \times S^2$, there are two ways to construct a consistent solution of $\mathbb{R}_t \times \mathbb{CP}^3$. One is to use the isomorphism $\mathbb{CP}^1 \simeq S^2$, and the other is to use the ansatz (2.31). Roughly said, the energy of the $\mathbb{CP}^1$ ("small") solution is just a half of the $\mathbb{RP}^2$ ("big") solution.

2.3.3   Giant magnons

The profile of $\mathbb{RP}^3$ dyonic giant magnons in conformal gauge is given by

$$t = 2\sqrt{1 + u_2^2}\, \tilde{\tau}, \quad z_1 = z_4^* = \frac{1}{\sqrt{2}} \xi_1, \quad z_2 = z_3^* = \frac{1}{\sqrt{2}} \xi_2, \tag{2.33}$$

where [69, 70]

$$\xi_1 = \frac{\sinh(X - i\omega)}{\cosh(X)} e^{i\tan(\omega)X + iu_1 T}, \qquad \xi_2 = \frac{\cos(\omega)}{\cosh(X)} e^{iu_2 T}, \tag{2.34}$$

$$T(\tau, \sigma) \equiv \frac{\tilde{\tau} - v\tilde{\sigma}}{\sqrt{1 - v^2}}, \quad X(\tau, \sigma) \equiv \frac{\tilde{\sigma} - v\tilde{\tau}}{\sqrt{1 - v^2}}, \quad v = \frac{\tan(\omega)}{u_1}, \quad u_1^2 - u_2^2 = \frac{1}{\cos^2\omega}. \tag{2.35}$$

6

The boundary conditions (2.32) are satisfied if $p = \pi - 2\omega$. The conserved charges are given by

$$E = 4h\,u_1 \left(1 - \frac{\tan^2(\omega)}{u_1^2}\right) \mathbf{K}(1),$$ (2.36)

$$J_1 = -J_4 = 4h\,u_1 \left[\left(1 - \frac{\tan^2(\omega)}{u_1^2}\right) \mathbf{K}(1) - \cos^2(\omega)\right],$$ (2.37)

$$J_2 = -J_3 = 4h\,u_2 \cos^2(\omega),$$ (2.38)

where $\mathbf{K}(1)$ is a divergent constant. They satisfy the relation

$$E - \frac{J_1 - J_4}{2} = \sqrt{\left(\frac{J_2 - J_3}{2}\right)^2 + 16h^2(\lambda)\sin^2\left(\frac{p}{2}\right)}.$$ (2.39)

It is interesting to consider semiclassical quantization of the $\mathbb{RP}^3$ dyonic giant magnons following [69]. If we observe the motion of the string (2.33) in the moving frame $(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) = \left(z_1\,e^{-it/2}, z_2, z_3, z_4\,e^{it/2}\right)$, it becomes periodic with respect to τ when $\omega = 0$, or $p = \pi$. This period measured by the AdS time t is $\mathcal{T} = 4\pi\coth\frac{q}{2}$, with q defined by $\sinh\frac{q}{2} \equiv u_2\cos(\omega)$. Then, the Bohr-Sommerfeld condition tells that the action variable

$$I = \frac{1}{2\pi}\oint \mathcal{T}\,d\left(E - \frac{J_1 - J_4}{2}\right)_{p=\pi} = J_2 - J_3,$$ (2.40)

is an integer.

Next, we consider $\mathbb{CP}^1$ giant magnon. We define polar coordinates on $\mathbb{CP}^1$ by

$$(z_1,\,z_2,\,z_3,\,z_4) = \left(\sin\frac{\theta}{2}\,e^{i\phi/2}, 0, 0, \cos\frac{\theta}{2}\,e^{-i\phi/2}\right).$$ (2.41)

The $\mathbb{CP}^1$ giant magnon is given by

$$\cos\theta = \frac{\cos(\omega)}{\cosh(X)}, \quad e^{i\phi} = \sqrt{\frac{\sinh(X - i\omega)}{\sinh(X + i\omega)}}\,e^{iX\tan(\omega)+iTu}, \quad u = \frac{1}{\cos(\omega)}.$$ (2.42)

We can rewrite the solution in terms of $\mathbb{CP}^1$ homogeneous coordinates. Such expressions are not unique due to the $U(1)$ degree of freedom. If we look for the solution with $A_X = 0$, we obtain

$$t = \frac{T + X\sin\omega}{\cos\omega}, \qquad z_1 = \frac{\sinh\left(\frac{X - i\omega}{2}\right)}{\sqrt{\cosh(X)}}\,e^{it/2}, \qquad z_4 = \frac{\cosh\left(\frac{X + i\omega}{2}\right)}{\sqrt{\cosh(X)}}\,e^{-it/2},$$ (2.43)

with $z_2 = z_3 = 0$. The $U(1)$ gauge fields defined in (2.4) are given by

$$A_X = 0, \qquad A_T = -\frac{1}{2\cosh X},$$ (2.44)

7

which is in fact the Lorenz gauge $\partial^\alpha A_\alpha = 0$. The Lagrangian density is related to the kink solution of sine-Gordon model by Pohlmeyer-Lund-Regge reduction [71, 72]

$$- 4 |D_\alpha z_i|^2 = 1 - \frac{2}{\cosh^2(X)} . \tag{2.45}$$

We must set $v = \sin\omega$ in (2.43) to obtain $t = \tau$. The boundary condition (2.25) gives $p = \pi/2 - \omega$. The conserved charges satisfy the dispersion relation

$$E - \frac{J_1 - J_4}{2} = 2h \left|\cos\omega\right| = 2h \left|\sin p\right| . \tag{2.46}$$

3 Dressing method

We review the paper of Sasaki [60], where he constructed solitons on the $\mathbb{CP}^N$ sigma model by dressing $SU(N+1)/U(N)$, and make a comment on the case of other embeddings.

3.1 Dressing $SU(N+1)/U(N)$

We begin with rewriting the equation of motion and the Bianchi identity in terms of a Lax pair, as

$$0 = \left\{\partial_+ - \frac{\partial_+ g \, g^{-1}}{1 + \lambda}\right\} \psi = \left\{\partial_+ - \frac{2\theta \left[\partial_+ P, P\right] \theta}{1 + \lambda}\right\} \psi,$$

$$0 = \left\{\partial_- - \frac{\partial_- g \, g^{-1}}{1 - \lambda}\right\} \psi = \left\{\partial_- - \frac{2\theta \left[\partial_- P, P\right] \theta}{1 - \lambda}\right\} \psi. \tag{3.1}$$

When $\lambda = 0$, they are solved by

$$\psi(\lambda = 0) = g = \theta \left(\mathbf{1}_{N+1} - 2P\right). \tag{3.2}$$

The relation $j_a = \theta \ell_a \theta$ imposes an additional constraint on $\psi(\lambda)$. If we rewrite (3.1) as[6]

$$0 = g^{-1} \left\{\partial_\pm - \frac{(\partial_\pm g) \, g^{-1}}{1 \pm \lambda}\right\} g \left(g^{-1}\psi\right) = \left\{\partial_\pm + \frac{g^{-1}(\partial_\pm g)}{1 \pm 1/\lambda}\right\} g^{-1}\psi = \theta \left\{\partial_\pm - \frac{(\partial_\pm g) \, g^{-1}}{1 \pm 1/\lambda}\right\} \theta g^{-1}\psi,$$

we find inversion symmetry

$$\psi(\lambda) = g\theta\psi(1/\lambda)\theta. \tag{3.3}$$

The unitarity condition on $\psi(\lambda)$ follows from (3.1), as

$$\left[\psi(\bar{\lambda})\right]^\dagger \psi(\lambda) = \mathbf{1}_{N+1} . \tag{3.4}$$

The right multiplication $\psi \to \psi U$ with a constant unitary matrix U leaves the system of equations (3.1) invariant. We can fix this ambiguity by $\psi(\lambda = \infty) = \mathbf{1}_{N+1}$.

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[6]The derivative, *i.e.* the first term in the bracket, acts on anything that follows.

A classical solution of the $\mathbb{CP}^N$ model is a map from worldsheet to $\mathbb{CP}^N$ subject to certain constraints. Since its image lies within $SU(N+1)$, all solutions are related by some unitary transformations. We assume that solutions are meromorphic functions of λ, and try to extend the unitary transformation over the complex λ plane. The dressing method provides us a simple way to construct such transformation matrices.

Let ψ be the simplest solution of (3.1), (3.3), (3.4), and $\tilde{\psi} = \chi\psi$ be another solution. We call χ the dressing matrix. If χ is not a constant matrix, then $\tilde{\psi}(\lambda = 0) = \tilde{g}$ becomes a new solution of the $\mathbb{CP}^N$ model.

The dressing matrix for the $\mathbb{CP}^N$ sigma model has been constructed explicitly in [60] for the case of Euclidean worldsheet. The dressing matrix for Minkowski worldsheet can be obtained by replacing $(\lambda_1, \bar{\lambda}_1)$ with $(\lambda_1, -\bar{\lambda}_1)$ in the Euclidean result, because the light-cone coordinates are not complex conjugate with each other in the Minkowski case.

Let ψ be the vacuum solution of (3.1). We introduce variables g and h by

$$\psi(\sigma^a, \lambda = 0) \equiv g = \theta\left(\mathbf{1} - 2P\right), \qquad \theta h = \psi(\sigma^a, \bar{\lambda}_1)u, \tag{3.5}$$

where u is a constant vector which parameterizes the dressed solution.[7] The dressing matrix of [60], modified for our case is given by

$$\chi(\lambda) = \mathbf{1} + \frac{Q_1}{\lambda - \lambda_1} + \frac{Q_2}{\lambda - 1/\lambda_1}, \tag{3.6}$$

$$Q_1 = \frac{\lambda_1}{\Lambda}\left(\frac{\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1}\theta h\beta h^\dagger\theta + \frac{1}{1 - \bar{\lambda}_1\lambda_1}gh\gamma h^\dagger\theta\right), \tag{3.7}$$

$$Q_2 = \frac{1}{\Lambda}\left(-\frac{1}{\lambda_1 - \bar{\lambda}_1}gh\beta h^\dagger g^\dagger + \frac{\bar{\lambda}_1}{1 - \bar{\lambda}_1\lambda_1}\theta h\gamma h^\dagger g^\dagger\right), \tag{3.8}$$

where Λ, β, γ are real numbers defined by

$$\Lambda = \bar{\lambda}_1\lambda_1\left(\frac{\beta^2}{(\lambda_1 - \bar{\lambda}_1)^2} - \frac{\gamma^2}{(1 - \bar{\lambda}_1\lambda_1)^2}\right), \quad \beta = h^\dagger h, \quad \gamma = h^\dagger\theta gh. \tag{3.9}$$

Since the dressed solution $\tilde{g} = \chi(0)g$ satisfies $(\theta\tilde{g})^2 = \mathbf{1}$ and $\tilde{g}\tilde{g}^\dagger = \mathbf{1}$, we can introduce the dressed projector by $\tilde{g} = \theta(\mathbf{1} - 2\tilde{P})$. The dressed projector takes the form

$$\tilde{P} = P - \frac{1}{2\Lambda}\Bigg\{\frac{\bar{\lambda}_1 h\beta h^\dagger(\mathbf{1} - 2P) - \lambda_1(\mathbf{1} - 2P)h\beta h^\dagger}{\lambda_1 - \bar{\lambda}_1}$$

$$+ \frac{(\mathbf{1} - 2P)h\gamma h^\dagger(\mathbf{1} - 2P) - \bar{\lambda}_1\lambda_1 h\gamma h^\dagger}{1 - \bar{\lambda}_1\lambda_1}\Bigg\}, \tag{3.10}$$

and its image, $\tilde{P}\tilde{z} = \tilde{z}$, is given by

$$\tilde{z} = \frac{1}{\Lambda_z}\left(\alpha z + hh^\dagger z\right), \quad \alpha = -\frac{\lambda_1\beta}{\lambda_1 - \bar{\lambda}_1} + \frac{\gamma}{1 - \bar{\lambda}_1\lambda_1}, \quad \Lambda_z = \sqrt{|\alpha|^2 + \frac{2\gamma}{1 - \bar{\lambda}_1\lambda_1}|h^\dagger z|^2}. \tag{3.11}$$

This gives us the dressed solution in terms of the normalized homogeneous coordinates of $\mathbb{CP}^N$.

---

[7]Is is possible to generalize u into a constant, rectangular matrix following [58, 59].

3.2 Dressed solution on $\mathbb{CP}^3$

Let us concentrate on the $\mathbb{CP}^3$ case. We choose vacuum as the following BPS solution:[8]

$$
t = \tau, \qquad z = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\tau/2} \\ 0 \\ 0 \\ e^{-i\tau/2} \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 0 & 0 & -e^{i\tau} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ e^{-i\tau} & 0 & 0 & 0 \end{pmatrix}. \tag{3.12}
$$

We parametrize the initial vectors u and h by

$$
u = \rho_u \begin{pmatrix} e^{i\nu_1} \cos \rho_1 \cos \rho_2 \\ e^{i\nu_2} \sin \rho_1 \cos \rho_3 \\ e^{i\nu_3} \sin \rho_1 \sin \rho_3 \\ e^{i\nu_4} \cos \rho_1 \sin \rho_2 \end{pmatrix}, \qquad h = \rho_u \begin{pmatrix} -e^{i\nu_4 + i\Sigma(\bar{\lambda}_1)} \cos \rho_1 \sin \rho_2 \\ e^{i\nu_2} \sin \rho_1 \cos \rho_3 \\ e^{i\nu_3} \sin \rho_1 \sin \rho_3 \\ -e^{i\nu_1 - i\Sigma(\bar{\lambda}_1)} \cos \rho_1 \cos \rho_2 \end{pmatrix}, \tag{3.13}
$$

$$
\Sigma(\lambda, \sigma^+, \sigma^-) = \frac{\sigma^+}{1 + \lambda} + \frac{\sigma^-}{1 - \lambda}, \tag{3.14}
$$

The parametrization (3.13) can be simplified. Multiplication by a complex constant $u \to cu$ does not modify the dressing matrix; we set $\rho_u = 1$, $\nu_1 + \nu_4 = 0$. Two real degrees of freedom of u go away by constant shifts of $\sigma^\pm$. Let Σ_0 be the displacement of $\Sigma(\bar{\lambda}_1, \sigma^+, \sigma^-)$ after such shifts. The translation $(\nu_1, \nu_4) \to (\nu_1 - \delta\nu, \nu_4 + \delta\nu)$ can be cancelled by the real part of Σ_0, and a particular combination of shifts on ρ_1, ρ_2, ρ_u is cancelled by the imaginary part; we may set $\nu \equiv \nu_4 - \nu_1 = 0$ and $\rho_2 = \pi/4$.

Under a global $SU(4)$ rotation, the vectors $z, h, \theta g h$ behave in the same manner:

$$
z \to Oz, \quad h \to Oh, \quad \theta g h \to O\theta g h, \quad \theta g \to O\theta g O^\dagger, \quad O \in SU(4), \tag{3.15}
$$

which leaves β and γ invariant. It has an $U(2)$ subgroup which acts trivially on z given in (3.12). We use this symmetry to fix $\nu_2 = \nu_3 = 0$ and $\rho_3 = 0$. Thus, the dressed solution on $\mathbb{CP}^3$ is reduced to the one on $\mathbb{CP}^2$.

With the new parametrization, u becomes

$$
u = \left(\frac{e^{-i\nu/2} \cos \rho_1}{\sqrt{2}}, \sin \rho_1, 0, \frac{e^{i\nu/2} \cos \rho_1}{\sqrt{2}} \right)^{\mathrm{T}}, \tag{3.16}
$$

and β, γ become

$$
\beta \equiv h^\dagger h \quad = \sin^2 \rho_1 + \cos^2 \rho_1 \cos \left[\Sigma(\lambda_1) - \Sigma(\bar{\lambda}_1) \right], \tag{3.17}
$$

$$
\gamma \equiv h^\dagger \theta g h = \sin^2 \rho_1 - \cos^2 \rho_1 \cos \left[\Sigma(\lambda_1) + \Sigma(\bar{\lambda}_1) - \tau + \nu \right], \tag{3.18}
$$

---

[8]Below we shall use (τ, σ) in place of $(\tilde{\tau}, \tilde{\sigma}) = (\mu\tau, \mu\sigma)$.

where we take care of ν-dependence for a later purpose.[9] If we rewrite the spectral parameters as $\lambda_1 = e^{(ip+q)/2}$ and $\bar{\lambda}_1 = e^{(-ip+q)/2}$, the worldsheet coordinate $\Sigma(\lambda_1)$ becomes

$$\Sigma(\lambda_1) = \frac{1}{2}\left(\tau - T\cos\alpha - iX\sin\alpha\right), \qquad T = \frac{\tau - v\sigma}{\sqrt{1 - v^2}}, \qquad X = \frac{\sigma - v\tau}{\sqrt{1 - v^2}}, \qquad (3.19)$$

where

$$v = \frac{\lambda_1 + \bar{\lambda}_1}{\bar{\lambda}_1\lambda_1 + 1} = \frac{\cos\left(\frac{p}{2}\right)}{\cosh\left(\frac{q}{2}\right)}, \qquad \tan\alpha = \frac{-i\left(\lambda_1 - \bar{\lambda}_1\right)}{\bar{\lambda}_1\lambda_1 - 1} = \frac{\sin\left(\frac{p}{2}\right)}{\sinh\left(\frac{q}{2}\right)}. \qquad (3.20)$$

3.2.1 The solution

The dressed solution is given by

$$\tilde{z}_1 = \frac{e^{i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[\cos^2\rho_1\left(-\frac{e^{X\sin\alpha}\lambda_1 + e^{-X\sin\alpha}\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1} + \frac{e^{-iT_\nu\cos\alpha}\left(e^{2iT_\nu\cos\alpha} + \lambda_1\bar{\lambda}_1\right)}{\lambda_1\bar{\lambda}_1 - 1}\right)\right.$$
$$\left. - \frac{2\sin^2\rho_1\left(\lambda_1^2 - 1\right)\bar{\lambda}_1}{(\lambda_1 - \bar{\lambda}_1)(\lambda_1\bar{\lambda}_1 - 1)}\right], \qquad (3.21)$$

$$\tilde{z}_2 = -\frac{\sin 2\rho_1}{2\Lambda_z}\cosh\left(\frac{X\sin\alpha - iT_\nu\cos\alpha}{2}\right), \qquad (3.22)$$

$$\tilde{z}_3 = 0, \qquad (3.23)$$

$$\tilde{z}_4 = \frac{e^{-i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[\cos^2\rho_1\left(-\frac{e^{-X\sin\alpha}\lambda_1 + e^{X\sin\alpha}\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1} + \frac{e^{-iT_\nu\cos\alpha}\left(1 + e^{2iT_\nu\cos\alpha}\lambda_1\bar{\lambda}_1\right)}{\lambda_1\bar{\lambda}_1 - 1}\right)\right.$$
$$\left. - \frac{2\sin^2\rho_1\left(\lambda_1^2 - 1\right)\bar{\lambda}_1}{(\lambda_1 - \bar{\lambda}_1)(\lambda_1\bar{\lambda}_1 - 1)}\right], \qquad (3.24)$$

$$\Lambda_z = \left\{\lambda_1\bar{\lambda}_1\left[\left(\frac{\sin^2\rho_1 - \cos^2\rho_1\cos(T_\nu\cos\alpha)}{\bar{\lambda}_1\lambda_1 - 1}\right)^2\right.\right.$$
$$\left.\left. - \left(\frac{\sin^2\rho_1 + \cos^2\rho_1\cosh(X\sin\alpha)}{\lambda_1 - \bar{\lambda}_1}\right)^2\right]\right\}^{1/2}, \qquad (3.25)$$

where $T_\nu\cos\alpha \equiv T\cos\alpha - \nu$. They satisfy the boundary conditions

$$\tilde{z}_1 \rightarrow \frac{i\,e^{i\tau/2}}{\sqrt{2}}\left(\frac{\lambda_1}{\bar{\lambda}_1}\right)^{\pm 1/2}, \quad \tilde{z}_2, \tilde{z}_3 \rightarrow 0, \quad \tilde{z}_4 \rightarrow \frac{i\,e^{-i\tau/2}}{\sqrt{2}}\left(\frac{\bar{\lambda}_1}{\lambda_1}\right)^{\pm 1/2}, \quad (\sigma \rightarrow \pm\infty). \qquad (3.26)$$

in complete agreement with (2.25). There is a symmetry

$$\tilde{z}_1\left(\frac{1}{\lambda_1}, \frac{1}{\bar{\lambda}_1}\right) = -\tilde{z}_4\left(\lambda_1, \bar{\lambda}_1\right) \quad \text{at} \ \ \tau = 0, \qquad \tilde{z}_2\left(\frac{1}{\lambda_1}, \frac{1}{\bar{\lambda}_1}\right) = \tilde{z}_2\left(\lambda_1, \bar{\lambda}_1\right), \qquad (3.27)$$

---

[9]See (3.38) and below.

11

which looks similar to the inversion symmetry of quasi-momenta discussed in [46]. The gauge fields and the Lagrangian density are given by

$$A_\tau = \frac{\cos^2 \rho_1 \cos\left(\frac{p}{2}\right) \cosh\left(\frac{q}{2}\right)}{2\Lambda_z^2 \left(\cosh q - \cos p\right)} \left(\frac{C_{X,+} \, \sin(T_\nu \cos\alpha)}{\sin\left(\frac{p}{2}\right) \cosh\left(\frac{q}{2}\right)} - \frac{C_{T,-} \, \sinh(X \sin\alpha)}{\cos\left(\frac{p}{2}\right) \sinh\left(\frac{q}{2}\right)} \right), \qquad (3.28)$$

$$A_\sigma = \frac{\cos^2 \rho_1 \cos\left(\frac{p}{2}\right) \cosh\left(\frac{q}{2}\right)}{2\Lambda_z^2 \left(\cosh q - \cos p\right)} \left(\frac{C_{T,-} \, \sinh(X \sin\alpha)}{\sinh\left(\frac{q}{2}\right) \cosh\left(\frac{q}{2}\right)} - \frac{C_{X,+} \, \sin(T_\nu \cos\alpha)}{\sin\left(\frac{p}{2}\right) \cos\left(\frac{p}{2}\right)} \right), \qquad (3.29)$$

$$|D_a z|^2 = -\frac{1}{32\Lambda_z^4 \left(\cosh q - \cos p\right)} \times$$

$$\left[\left(\frac{C_{X,+}}{\sin\left(\frac{p}{2}\right)} \right)^2 \left(2\sin^4 \rho_1 - 6\cos^4 \rho_1 \cos^2(T_\nu \cos\alpha) + C_{X,-}^2 + \frac{C_{X,+}^2 \, \sinh^2\left(\frac{q}{2}\right)}{\sin^2\left(\frac{p}{2}\right)} \right) \right.$$

$$\left. + \left(\frac{C_{T,-}}{\sinh\left(\frac{q}{2}\right)} \right)^2 \left(2\sin^4 \rho_1 - 6\cos^4 \rho_1 \cosh^2(X \sin\alpha) + C_{T,+}^2 + \frac{C_{T,-}^2 \, \sin^2\left(\frac{p}{2}\right)}{\sinh^2\left(\frac{q}{2}\right)} \right) \right], \quad (3.30)$$

where

$$C_{T,\pm} \equiv \cos(T_\nu \cos\alpha) \cos^2 \rho_1 \pm \sin^2 \rho_1 , \quad C_{X,\pm} \equiv \cosh(X \sin\alpha) \cos^2 \rho_1 \pm \sin^2 \rho_1 . \qquad (3.31)$$

They satisfy Lorenz gauge condition $\partial^\alpha A_\alpha = 0$. As discussed in Appendix A, the expressions (3.28)-(3.30) defines breather-like solutions of $SU(3)/U(2)$ symmetric space sine-Gordon model, as shown in Figure 1.
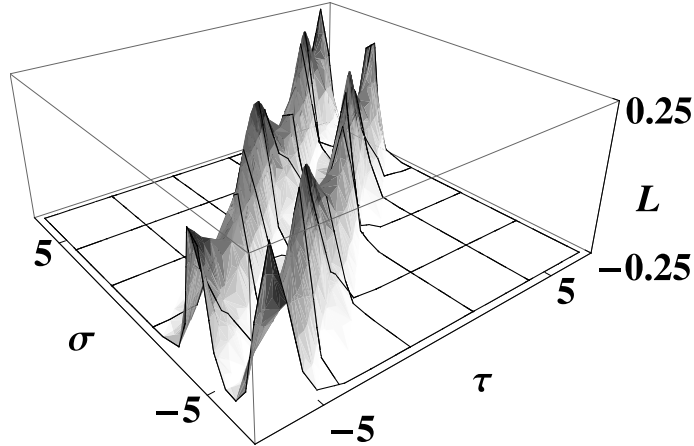

Figure 1: Lagrangian density $L = |D_\alpha z|^2$ for $\rho_1 = 0.9$, $\lambda_1 = \exp\left(0.6i + 0.05\right)$. This figure describes a breather-like solution of $SU(3)/U(2)$ symmetric space sine-Gordon model.

Since the gauge fields are odd under parity transformation $(X, T_\nu) \to (-X, -T_\nu)$, they do not contribute to the conserved charge (2.10). The conserved charges are evaluated as

$$E - \frac{J_1 - J_4}{2} = 4h \sin \frac{p}{2} \cosh \frac{q}{2} = \frac{h}{i} \left(\lambda_1 - \frac{1}{\lambda_1} - \bar{\lambda}_1 + \frac{1}{\bar{\lambda}_1} \right), \quad J_1 + J_4 = J_2 = J_3 = 0, \quad (3.32)$$

which are independent of ρ_1. On the Chern-Simons side, these modes should correspond to excitations like

$$Y^1(Y_4)^\dagger \quad \to \quad Y^1(Y_1)^\dagger, \ Y^2(Y_2)^\dagger, \ Y^3(Y_3)^\dagger, \ Y^4(Y_4)^\dagger. \quad (3.33)$$

Let us rewrite the dispersion relation (3.32) as

$$E - \frac{J_1 - J_4}{2} = \sqrt{n^2 + 16h^2 \sin^2 \frac{p}{2}}, \quad n \equiv 4h \sin \frac{p}{2} \sinh \frac{q}{2}. \quad (3.34)$$

By repeating the argument we did in (2.40), one can show that n is an action variable semiclassically quantized to integer. This situation is same as that of breather-like giant magnon solutions discussed in [45], so let us recall what we have learned in the $\text{AdS}_5 \times \text{S}^5$ case. In the paper [45], they rewrote the energy of a breather-like solution as the energy of a pair of elementary solutions, as

$$2h' \sin \frac{p_1}{2} + 2h' \sin \frac{p_2}{2} = \sqrt{n'^2 + 16h'^2 \sin^2 \frac{p}{2}}, \quad n' \equiv 4h' \sin \frac{p}{2} \sinh \frac{q}{2}, \quad h' \equiv \frac{\sqrt{\lambda}}{2\pi}, \quad (3.35)$$

with $p_1 = p + iq$, $p_2 = p - iq$, and n' an action variable. This relation suggests that the semiclassical S-matrix should have single poles (or zeroes) at

$$\cos \left(\frac{p_1}{2} \right) - \cos \left(\frac{p_2}{2} \right) = -\frac{in'}{4h'} \quad (n' \in \mathbb{Z}_{\geq 1}). \quad (3.36)$$

However, the full quantum theory of $\text{AdS}_5 \times \text{S}^5$ string predicts the existence of double poles at

$$u(p_1) - u(p_2) = -\frac{in'}{2h'} \quad (n' \in \mathbb{Z}_{>1}), \qquad u(p) \equiv \frac{1}{2h'} \cot \frac{p}{2} \sqrt{1 + 4h'^2 \sin^2 \frac{p}{2}}, \quad (3.37)$$

where the branch of square root is carefully chosen [73, 74, 75]. From this lesson, we may think of (3.34) as giving the mean density of poles or zeroes (but not their exact location) for the S-matrix of $\text{AdS}_4 \times \mathbb{CP}^3$ string theory.[10]

3.2.2 Taking limits

The dressed solution contains three real parameters $\text{Re}\,\lambda_1$, $\text{Im}\,\lambda_1$ and ρ_1. The dressing matrix always becomes trivial in the limit $\text{Im}\,\lambda_1 \to 0$, or $\rho_1 \to \pi/2$. So we concentrate on two other interesting limits, $|\lambda_1| \to 1$ and $\rho_1 \to 0$.

---

[10]The author acknowledges to the referee of JHEP a cautious remark on this point.

As can be seen from (3.7) and (3.8), the limit $|\lambda_1| \to 1$, or equivalently $q \to 0$, forces the dressing matrix to be trivial unless $\gamma = 0$. Hence we send $|\lambda_1|$ to 1 under the constraint $\gamma = 0$, as explained below. First, we see from (3.18) that if $q = 0$, the condition $\gamma = 0$ is equivalent to

$$\gamma \equiv \sin^2 \rho_1 - \cos^2 \rho_1 \cos\left(T_\nu \cos\alpha\right) = \sin^2 \rho_1 - \cos^2 \rho_1 \cos\nu = 0. \tag{3.38}$$

We assume $|\rho_1| \le \pi/4$ so that this equation has solutions with real ν. Second, we expand the numerator of possibly divergent terms around $q = 0$. This can be done by using

$$\nu = \arccos\left(\tan^2 \rho_1\right) + q\,\delta\nu + \mathcal{O}(q^2), \tag{3.39}$$

$$e^{-iT_\nu \cos\alpha} = -\frac{\left(\sqrt{\cos(2\rho_1)} - i\right)^2}{2\cos^2(\rho_1)} \left\{1 - iq\left(\frac{T}{2\sin\frac{p}{2}} - \delta\nu\right) + \mathcal{O}(q^2)\right\}. \tag{3.40}$$

We can set $\delta\nu = 0$ by an appropriate shift of T.

There is no difficulty in taking the limit $\rho_1 \to 0$, and the $\mathbb{CP}^2$ solution reduces to that of $\mathbb{CP}^1$.

(1) $\mathbb{CP}^1$ limit

When $\rho_1 = 0$, the dressed solution becomes

$$\tilde{z}_1 = \frac{e^{i\tau/2}}{2\sqrt{2}\,\Lambda_z} \left[-\frac{e^{X\sin\alpha}\lambda_1 + e^{-X\sin\alpha}\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1} + \frac{e^{-iT_\nu\cos\alpha}\left(e^{2iT_\nu\cos\alpha} + \lambda_1\bar{\lambda}_1\right)}{\lambda_1\bar{\lambda}_1 - 1}\right], \tag{3.41}$$

$$\tilde{z}_4 = \frac{e^{-i\tau/2}}{2\sqrt{2}\,\Lambda_z} \left[-\frac{e^{-X\sin\alpha}\lambda_1 + e^{X\sin\alpha}\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1} + \frac{e^{-iT_\nu\cos\alpha}\left(1 + e^{2iT_\nu\cos\alpha}\lambda_1\bar{\lambda}_1\right)}{\lambda_1\bar{\lambda}_1 - 1}\right], \tag{3.42}$$

$$\Lambda_z = \left\{\lambda_1\bar{\lambda}_1 \left[\left(\frac{\cos(T_\nu\cos\alpha)}{\bar{\lambda}_1\lambda_1 - 1}\right)^2 - \left(\frac{\cosh(X\sin\alpha)}{\lambda_1 - \bar{\lambda}_1}\right)^2\right]\right\}^{1/2}. \tag{3.43}$$

with $\tilde{z}_2 = \tilde{z}_3 = 0$. They satisfy $z_1 = (z_4)^*$.

If we drop all T_ν-dependent terms in the solution (3.41)-(3.42) by hand (rather than taking a limit) and set $|\lambda_1| = 1$, we obtain

$$\frac{\tilde{z}_1}{\tilde{z}_4} = e^{i\tau}\,\frac{\cosh\left(X + \frac{ip}{2}\right)}{\cosh\left(X - \frac{ip}{2}\right)} = e^{i\tilde{\tau}}\,\frac{\sinh\left(\frac{\tilde{X} - i\tilde{\omega}}{2}\right)}{\cosh\left(\frac{\tilde{X} + i\tilde{\omega}}{2}\right)} \quad \tilde{\tau} = \tau + \frac{\pi}{2},\ \tilde{X} = \frac{X}{2} + \frac{i\pi}{4},\ \tilde{\omega} = \frac{\pi}{2} - p, \tag{3.44}$$

which coincides with the profile of $\mathbb{CP}^1$ giant magnons given in (2.43) with $\omega = \tilde{\omega}$.[11] Although both solutions, (3.41)-(3.43) and (2.43), have the pole at the same position $\lambda = \lambda_1,\,\bar{\lambda}_1$ on the unit circle, the former carries twice as large charges as the latter.

---

[11]The relation between ω and p is actually incorrect, so (3.44) is not a consistent solution. This is because the coordinate $\tilde{X}$ of (3.44) is boosted by $v = \cos\frac{p}{2}$, while X in (2.43) is by $v = \sin\omega$.

(2) The limit $|\lambda_1| \to 1$

We carefully take the limit $|\lambda_1| \to 1$ for general values of ρ_1 as described above. The string profile becomes

$$\tilde{z}_1 = \frac{e^{i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[-\cos^2\rho_1\left(\frac{e^X\lambda_1 + e^{-X}\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1}\right)\right.$$

$$\left. + i\sqrt{\cos(2\rho_1)}\left(1 + \frac{2T}{\lambda_1 - \bar{\lambda}_1}\right) - \sin^2\rho_1\left(\frac{\lambda_1 + \bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1}\right)\right], \quad (3.45)$$

$$\tilde{z}_2 = -\frac{\sin\rho_1}{\sqrt{2}\Lambda_z}\left[\cosh\left(\frac{X}{2}\right) + i\sqrt{\cos(2\rho_1)}\,\sinh\left(\frac{X}{2}\right)\right], \quad (3.46)$$

$$\tilde{z}_3 = 0, \quad (3.47)$$

$$\tilde{z}_4 = \frac{e^{-i\tau/2}}{2\sqrt{2}\,\Lambda_z}\left[-\cos^2\rho_1\left(\frac{e^{-X}\lambda_1 + e^X\bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1}\right)\right.$$

$$\left. - i\sqrt{\cos(2\rho_1)}\left(1 - \frac{2T}{\lambda_1 - \bar{\lambda}_1}\right) - \sin^2\rho_1\left(\frac{\lambda_1 + \bar{\lambda}_1}{\lambda_1 - \bar{\lambda}_1}\right)\right], \quad (3.48)$$

$$\Lambda_z = \left\{\frac{-1}{(\lambda_1 - \bar{\lambda}_1)^2}\left[\left(\cos^2\rho_1\cosh X + \sin^2\rho_1\right)^2 + \cos 2\rho_1\,T^2\right]\right\}^{1/2}. \quad (3.49)$$

The gauge fields and the Lagrangian density are given by

$$A_\tau = -\frac{\sqrt{\cos(2\rho_1)}}{4\sin^3\left(\frac{p}{2}\right)\Lambda_z^2}\left[T\sinh(X)\cos^2(\rho_1) + \cos\left(\frac{p}{2}\right)\left(\cosh(X)\cos^2(\rho_1) + \sin^2(\rho_1)\right)\right], \quad (3.50)$$

$$A_\sigma = \frac{\sqrt{\cos(2\rho_1)}}{4\sin^3\left(\frac{p}{2}\right)\Lambda_z^2}\left[T\sinh(X)\cos^2(\rho_1)\cos\left(\frac{p}{2}\right) + \cosh(X)\cos^2(\rho_1) + \sin^2(\rho_1)\right], \quad (3.51)$$

$$|D_a z|^2 = \frac{-1}{64\sin^4\left(\frac{p}{2}\right)\Lambda_z^4}\left[T^4\cos^2(2\rho_1) - 6T^2\cos(2\rho_1)\left(\cos^4(\rho_1)\cosh^2(X) - \sin^4(\rho_1)\right)\right.$$

$$\left. + \left(\cos^2(\rho_1)\cosh(X) - 3\sin^2(\rho_1)\right)\left(\cosh(X)\cos^2(\rho_1) + \sin^2(\rho_1)\right)^3\right]. \quad (3.52)$$

They satisfy the Lorenz gauge condition, $\partial^\alpha A_\alpha = 0$. Note that the gauge fields are even under parity transformation.

Since the gauge fields are proportional to $\sqrt{\cos(2\rho_1)}$, the parameter ρ_1 must take values in between $[-\pi/4, \pi/4]$, as is expected from the condition (3.38). When $\rho_1 = \pm\pi/4$, they become equivalent to the $\mathbb{RP}^2$ giant magnon solutions (2.33), after rescaling of (τ, σ). In terms of the reduced sine-Gordon system, it implies that the breather of a sine-Gordon model ($\rho_1 = 0$) is continuously connected to the kink of another sine-Gordon model ($\rho_1 = \pi/4$, $|\lambda_1| = 1$).

15

3.3 Comments on other embeddings

As noticed in [63, 64], the dressings of different coset spaces give us different soliton solutions. For the present case, the $\mathbb{RP}^3$ dyonic giant magnons can be easily obtained by dressing $SU(2) \times \overline{SU(2)}$. The dressing of the BPS vacuum on $SU(4)/U(3)$ with rank one projector cannot reproduce such solutions.[12]

We may consider dressing the $SO(6)/U(3)$ model instead of $SU(4)/U(3)$. As discussed in Appendix B, however, it turns out that the dressing matrix becomes trivial when the spectral parameters approach the unit circle. We are unable to construct "dyonic giant magnons" on $\mathbb{CP}^3$ neither in this way.

4 Summary and Discussion

In this paper, we consider the classical string spectrum of $\mathbb{R}_t \times \mathbb{CP}^3$ sigma model in the decompactification limit. We constructed a family of giant magnon solutions with $SU(2)$ symmetry by means of the dressing method on $SU(4)/U(3)$. All such solutions obey the same square-root type dispersion relation which is, at least naïvely, expected from the BPS relation of the centrally extended $psu(2|2)$ symmetry.

The new solutions are neutral with respect to the global charges of $psu(2|2)$, and thus they could be non-BPS boundstates which receive quantum corrections. It is known that there are no non-BPS boundstates in the $\text{AdS}_5 \times \text{S}^5$ case, in the sense that neutral states are equivalent to a composite of two oppositely-charged dyonic giant magnons [63]. Since different boundstate spectrum should lead to different singularity structure of the worldsheet S-matrix [75, 76], it is interesting to determine whether our solutions are BPS or not.

There remains a problem to construct an explicit profile of dyonic giant magnon solutions. We expect to find ways to construct such soliton solutions with the help of classical integrability. One direction is to study the reduced sine-Gordon system discussed in Appendix A. Another direction is to study in detail classical membrane spectrum in $\text{AdS}_4 \times \text{S}^7/\mathbb{Z}_k$ for general k, and carefully take the limit $k \to \infty$ limit [77, 78, 79, 80]. We hope to revisit such problems in future.

Acknowledgments

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[12]The term "rank" refers to the rank of constant vector u. See footnote 7.

A Pohlmeyer reduction on $\mathbb{CP}^N$

We revisit the reduction problem of classical strings on $\mathbb{R}_t \times \mathbb{CP}^N$. It is known that equations of motion on $\mathbb{CP}^N \simeq SU(N+1)/U(N)$ can equivalently be rewritten as symmetric space sine-Gordon equations [81, 82, 83, 84, 85, 86]. We would like to clarify an explicit relation between the $\mathbb{CP}^N$ coordinates and the sine-Gordon fields in order to relate the solutions of two theories. They will also help us to construct new solutions of the $\mathbb{CP}^N$ sigma model as in [69, 70].

The reduction procedure goes parallel with the $\mathbb{R}_t \times S^N$ case done by Pohlmeyer, which can be outlined as follows. One chooses a $U(1)$-invariant, orthonormal basis of the tangent space of $\mathbb{C}^{N+1}$, say $\{\vec{v}^1, \cdots, \vec{v}^{N+1}\}$. Then one differentiate the basis vectors. The result can be expanded by the basis itself as $\partial_\alpha \vec{v}^k = M_\alpha \cdot \vec{v}^k$. The compatibility condition $(D_\alpha D_\beta - D_\beta D_\alpha)\, \vec{v}^k = -i F_{\alpha\beta}\, \vec{v}^k$ gives differential equations for the matrix elements of M_α. One can recast them into sine-Gordon like equations through appropriate parametrization of M_α.

A.1 Constraints and Identities

Let us define light-cone coordinates by $\partial_\pm = \partial_\tau \pm \partial_\sigma$. The energy-momentum conservation becomes $\partial_+ T_{--} = \partial_- T_{++} = 0$, and we can rewrite Virasoro constraints as

$$\frac{1}{4} = |D_+ z_i|^2 = |D_- z_i|^2\,, \tag{A.1}$$

where we rescaled worldsheet coordinates to set $\kappa = 1$. We introduce a dynamical degree of freedom u and rewrite the equations of motion as

$$\frac{\cos u}{2} = \overline{D_+ z_i}\, D_- z_i + D_+ z_i\, \overline{D_- z_i}\,, \tag{A.2}$$

$$-\frac{\cos u}{2}\, z_i = D_+ D_- z_i + D_- D_+ z_i\,. \tag{A.3}$$

We can derive some identities from $\bar{z}_i\, z_i = 1$ and (2.4),

$$0 = \bar{z}_i\, D_\alpha z_i\,, \tag{A.4}$$

$$F_{+-} = -i\left(\overline{D_+ z_i}\, D_- z_i - D_+ z_i\, \overline{D_- z_i}\right)\,. \tag{A.5}$$

By combining (A.5) with (A.2), we find

$$\overline{D_+ z_i}\, D_- z_i = \frac{\cos u + 2i F_{+-}}{4}\,, \qquad \overline{D_- z_i}\, D_+ z_i = \frac{\cos u - 2i F_{+-}}{4}\,, \tag{A.6}$$

and (A.5) with (A.3),

$$D_+ D_- z_i = -\left(\frac{\cos u + 2i F_{+-}}{4}\right) z_i\,. \tag{A.7}$$

We differentiate the above equations to find identity relations. Derivative of (A.4) gives

$$-\frac{1}{4} = \bar{z}_i\, D_+^2 z_i = \bar{z}_i\, D_-^2 z_i\,. \tag{A.8}$$

17

Derivative of Virasoro constraints (A.1) gives

$$0 = \overline{D_\pm^2 z_i}\, D_\pm z_i + D_\pm^2 z_i\, \overline{D_\pm z_i}\,. \tag{A.9}$$

From (A.6), we obtain

$$\overline{D_+ z_i}\, D_-^2 z_i = \partial_- \left(\frac{\cos u + 2iF_{+-}}{4} \right), \qquad \overline{D_- z_i}\, D_+^2 z_i = \partial_+ \left(\frac{\cos u - 2iF_{+-}}{4} \right), \tag{A.10}$$

and

$$4\overline{D_+^2 z_i}\, D_-^2 z_i = \partial_+ \partial_- \left(\cos u + 2iF_{+-} \right) + \frac{1}{4}\left(\cos u + 2iF_{+-} \right)\left(\cos u + 6iF_{+-} \right). \tag{A.11}$$

We introduce another dynamical degrees of freedom $H_\pm$ by

$$H_\pm \equiv -i \left(\overline{D_\pm^2 z_i}\, D_\pm z_i - D_\pm^2 z_i\, \overline{D_\pm z_i} \right), \tag{A.12}$$

so that we have

$$\overline{D_\pm z_i}\, D_\pm^2 z_i = -\frac{i}{2}\, H_\pm\,, \tag{A.13}$$

where we used (A.9). It follows that

$$\partial_+ H_- = -\partial_- H_+ = F_{+-}\,. \tag{A.14}$$

Thus, $H_\pm/2$ coincides with the $U(1)$ gauge field $A_\pm$ satisfying Lorenz gauge condition $\partial^\alpha A_\alpha = 0$. Let us introduce a new variable φ by

$$H_\pm \equiv \mp \partial_\pm \varphi, \qquad \partial_+ \partial_- \varphi = F_{+-}\,. \tag{A.15}$$

A.2 Reduction procedure

By using $\mathbb{CP}^N \subset \mathbb{C}^{N+1}$, we expand the second-order covariant derivatives as[13]

$$D_+^2 z = a_1\, z + a_2\, D_+ z + a_3\, D_- z + \sum_{j=4}^{N+1} a_j\, v^j\,,$$
$$D_-^2 z = b_1\, z + b_2\, D_- z + b_3\, D_+ z + \sum_{j=4}^{N+1} b_j\, v^j\,, \tag{A.16}$$

where v^j are gauge-invariant basis vectors, satisfying orthonormal conditions

$$0 = \bar{v}^j z = \bar{v}^j D_\pm z\,, \qquad \delta^{jk} = \bar{v}^j\, v^k. \tag{A.17}$$

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[13]We omit the subscript i from z_i below.

The coefficients a_1 and b_1 are determined by the equations (A.8) as

$$a_1 = b_1 = -\frac{1}{4}. \tag{A.18}$$

The coefficients a_2, a_3 are constrained by the equations (A.10) and (A.13) as

$$\begin{pmatrix} \cos u - 2iF_{+-} & 1 \\ 1 & \cos u + 2iF_{+-} \end{pmatrix} \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \partial_+ \left(\cos u - 2iF_{+-} \right) \\ 2i\partial_+\varphi \end{pmatrix}. \tag{A.19}$$

Constraints for b_2, b_3 are given by

$$\begin{pmatrix} \cos u + 2iF_{+-} & 1 \\ 1 & \cos u - 2iF_{+-} \end{pmatrix} \begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \partial_- \left(\cos u + 2iF_{+-} \right) \\ -2i\partial_-\varphi \end{pmatrix}. \tag{A.20}$$

If the 2×2 matrix in (A.19) is not degenerate, that is $F_{+-} \neq \pm \sin(u)/2$, then this equation is solved by

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \frac{1}{\bar\eta \eta - 1} \begin{pmatrix} \eta & -1 \\ -1 & \bar\eta \end{pmatrix} \begin{pmatrix} \partial_+\bar\eta \\ 2i\partial_+\varphi \end{pmatrix}, \tag{A.21}$$

where $\eta = \cos u + 2iF_{+-}$ and $\bar\eta = \cos u - 2iF_{+-}$. We can evaluate the left hand side of (A.11) as

$$4\overline{D_+^2 z_i}\, D_-^2 z_i = 4\bar a_1 b_1 + \bar a_2 b_3 + \bar a_3 b_2 + \eta\, \bar a_2 b_2 + \bar\eta\, \bar a_3 b_3 + 4\sum_{j=4}^{N+1} \bar a_j b_j, \tag{A.22}$$

$$= \frac{1}{4} + \frac{\bar\eta\, \partial_+\eta\, \partial_-\eta - 4\eta\, \partial_+\varphi\, \partial_-\varphi + 2i \left(\partial_-\eta\, \partial_+\varphi + \partial_+\eta\, \partial_-\varphi \right)}{\bar\eta\eta - 1} + 4\sum_{j=4}^{N+1} \bar a_j b_j. \tag{A.23}$$

Suppose now that the 2×2 matrix in (A.19) is degenerate. We may set $F_{+-} = \sin(u)/2$, by the flip of $u \mapsto -u$ if necessary. The equation (A.19) tells us

$$a_3 = -\left(a_2 + i\partial_+ u \right) e^{-iu}, \qquad \partial_+\varphi = -\frac{\partial_+ u}{2}, \tag{A.24}$$

$$b_3 = -\left(b_2 - i\partial_- u \right) e^{iu}, \qquad \partial_-\varphi = -\frac{\partial_- u}{2}. \tag{A.25}$$

The condition (A.14) gives

$$0 = -\partial_+\partial_- u - \sin u = \partial_a^2 u - \sin u, \tag{A.26}$$

which is sine-Gordon equation.

For our reduction procedure to be consistent, the relation (A.11) must reduce to the same sine-Gordon equation as (A.26). This can be checked by evaluating (A.22) by using (A.24) and (A.25). The result is

$$4\overline{D_+^2 z_i}\, D_-^2 z_i = \frac{1}{4} - e^{iu}\, \partial_+ u\, \partial_- u + 4\sum_{j=4}^{N+1} \bar a_j b_j, \tag{A.27}$$

19

which is independent of undetermined coefficients a_2 and b_2. We will see later that the coefficients a_j, b_j $(j \geq 4)$ vanish in this case. Thus the second-order differential equation (A.11) is equivalent to (A.26).

A.3 $\mathbb{CP}^1$ case

Recall that $\mathbb{CP}^1$ is locally isomorphic to S^2, and the Pohlmeyer reduction of $\mathbb{R}_t \times S^2$ sigma model provides us the sine-Gordon equation. From this reasoning we can fix the normalization of sine-Gordon coupling. We find

$$- \cos u = 4 \left| D_\alpha z_i \right|^2, \tag{A.28}$$

where $u(\tau, \sigma)$ satisfies the sine-Gordon equation $\partial_\alpha^2 u - \sin u = 0$.

This equation has the same normalization as the one obtained for the degenerate case (A.26). This result is expected. Since the tangent space of $\mathbb{C}^2$ is two-dimensional, the set of equations (A.19) must be overdetermined and the expansion (A.16) must contain a redundant parameter. In fact, the degeneracy condition $F_{+-} = \pm \sin(u)/2$ is identically satisfied on the $\mathbb{CP}^1$ space.

A.4 Nondegenerate $\mathbb{CP}^2$ case

Since the tangent space of $\mathbb{C}^3$ is three-dimensional, we can set $a_j = b_j = 0$ for $j \geq 4$ in (A.23). The differential equation (A.11) becomes

$$- \partial_+ \partial_- \eta - \frac{\bar{\eta} \, \partial_+ \eta \, \partial_- \eta - 4\eta \, \partial_+ \varphi \, \partial_- \varphi + 2i \left(\partial_- \eta \, \partial_+ \varphi + \partial_+ \eta \, \partial_- \varphi \right)}{1 - \bar{\eta} \eta} + \frac{1 - 2\eta^2 + \bar{\eta} \eta}{4} = 0. \tag{A.29}$$

We redefine $\eta = \zeta \, e^{-2i\varphi}$ and $\bar{\eta} = \bar{\zeta} \, e^{2i\varphi}$, and rewrite this equation by using

$$- 4i\partial_a^2 \varphi = \eta - \bar{\eta} = \zeta \, e^{-2i\varphi} - \bar{\zeta} \, e^{2i\varphi}, \tag{A.30}$$

as

$$\partial_a^2 \zeta + \frac{\bar{\zeta} (\partial_a \zeta)^2}{1 - \bar{\zeta} \zeta} + \left(\frac{1 - \bar{\zeta} \zeta}{4} \right) e^{2i\varphi} = 0. \tag{A.31}$$

All of the equations (A.30), (A.31), and the complex conjugate of (A.31), can be derived from the Lagrangian

$$\mathcal{L} = \frac{\partial^a \bar{\zeta} \partial_a \zeta}{1 - \bar{\zeta} \zeta} - \frac{\bar{\zeta} \, e^{2i\varphi} + \zeta \, e^{-2i\varphi}}{4} + (\partial_a \varphi)^2. \tag{A.32}$$

One can reproduce the sine-Gordon equation for $\mathbb{CP}^1$ by setting $\zeta = 1$. This solves the equation (A.31), and the constraint (A.30) gives

$$0 = \partial_a^2 \varphi - \frac{1}{2} \sin 2\varphi, \tag{A.33}$$

20

which is (A.26) with $u = -2\varphi$. If we set $\varphi = 0$, that is $\bar{\zeta} = \zeta = \cos u$, the equation (A.31) becomes

$$\partial_a^2 \zeta + \frac{\zeta (\partial_a \zeta)^2}{1 - \zeta^2} + \frac{1 - \zeta^2}{4} = 0, \tag{A.34}$$

which is again sine-Gordon, with the different normalization from (A.33).

If we define three real variables α, β, γ by

$$\eta = -\cos \alpha \, e^{i\gamma}, \qquad \varphi = \frac{\beta}{2}, \qquad \zeta = \eta \, e^{2i\varphi} = -\cos \alpha \, e^{i(\gamma+\beta)}, \tag{A.35}$$

we can rewrite (A.32) as

$$\mathcal{L} = (\partial_a \alpha)^2 + \cot^2 \alpha \, \partial_a (\gamma + \beta) \, \partial^a (\gamma + \beta) + \frac{1}{2} \cos \alpha \cos \gamma + \frac{1}{4} (\partial_a \beta)^2, \tag{A.36}$$

which is the Lagrangian obtained in [83]. The explicit relation between sine-Gordon fields α, β, γ and $\mathbb{CP}^2$ coordinates has been mentioned in [83], which agree with ours (A.6), (A.13) up to $\gamma \to \gamma + \pi$.

A.5 Degenerate $\mathbb{CP}^N$ cases

We will show that the coefficients a_j, b_j $(j \geq 4)$ always vanish in the degenerate case. Let us expand the second-order covariant derivatives as

$$D_+^2 z_i = a_1 z_i + a_2 D_+ z_i + a_3 D_- z_i + \sum_{j=4}^{N+1} a_j v_i^{(j)},$$

$$D_-^2 z_i = b_1 z_i + b_2 D_- z_i + b_3 D_+ z_i + \sum_{j=4}^{N+1} b_j v_i^{(j)}, \tag{A.37}$$

where $v^{(j)}$ satisfy the orthonormal conditions (A.17). We omit the indices i below.

By taking derivatives of (A.17) and using identities in Section A.1, we obtain

$$\overline{D_\alpha v^{(j)}} \, z = 0, \qquad \bar{v}^j D_\alpha D_\beta z + \overline{D_\alpha v^j} \, D_\beta z = 0. \tag{A.38}$$

We can rewrite the coefficients a_j and b_j as

$$a_j = \bar{v}^j D_+^2 z = -\overline{D_+ v^j} \, D_+ z, \qquad b_j = \bar{v}^j D_-^2 z = -\overline{D_- v^j} \, D_- z. \tag{A.39}$$

We also find

$$\overline{D_\pm v^j} \, D_\mp z = -\bar{v}^j D_\pm D_\mp z = \left(\frac{\cos u + 2i F_{+-}}{4} \right) \bar{v}^j z = 0. \tag{A.40}$$

Let us expand the covariant derivatives of the basis vectors as

$$\overline{D_+ v} = r_1 \bar{z} + r_2 \overline{D_+ z} + r_3 \overline{D_- z} + \sum_{j=4}^{N+1} r_j \bar{v}^j,$$

$$\overline{D_- v} = s_1 \bar{z} + s_2 \overline{D_- z} + s_3 \overline{D_+ z} + \sum_{j=4}^{N+1} s_j \bar{v}^j. \tag{A.41}$$

21

The first equation of (A.38) shows $r_1 = s_1 = 0$. The relations (A.39) and (A.40) give us

$$\begin{pmatrix} 1 & \cos u + 2iF_{+-} \\ \cos u - 2iF_{+-} & 1 \end{pmatrix} \begin{pmatrix} r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -a_j \\ 0 \end{pmatrix}. \tag{A.42}$$

and similar equations for b_j. When this 2×2 matrix is degenerate, that is when $F_{+-} = \pm \sin(u)/2$, we obtain $a_j = b_j = 0$ for $j \geq 4$.

For the non-degenerate case, one can derive differential equations for a_j and b_j from the compatibility condition $(D_+ D_- - D_- D_+) v^j = -iF_{+-} v^j$. We do not discuss them here because they look complicated. We just refer to [87], which studied the reduced sine-Gordon type equations of $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

B   On dressing $SO(6)/U(3)$

We will show that the dressing matrix of the $SO(6)/U(3)$ model becomes trivial when the spectral parameters are on the unit circle following [58, 59].

In the $SO(6)/U(3)$ model, the minimum set of poles in the dressing matrix is λ_1, $1/\lambda_1$, $\bar{\lambda}_1$, $1/\bar{\lambda}_1$ when $|\lambda_1| \neq 1$. We write the dressing matrix χ and χ^{-1} as

$$\chi(\lambda) = \mathbf{1}_6 + \frac{Q_1}{\lambda - \lambda_1} + \frac{Q_{\hat{1}}}{\lambda - 1/\lambda_1} + \frac{Q_{\bar{1}}}{\lambda - \bar{\lambda}_1} + \frac{Q_{\hat{\bar{1}}}}{\lambda - 1/\bar{\lambda}_1}, \tag{B.1}$$

$$\chi^{-1}(\lambda) = \mathbf{1}_6 + \frac{R_1}{\lambda - \lambda_1} + \frac{R_{\hat{1}}}{\lambda - 1/\lambda_1} + \frac{R_{\bar{1}}}{\lambda - \bar{\lambda}_1} + \frac{R_{\hat{\bar{1}}}}{\lambda - 1/\bar{\lambda}_1}, \tag{B.2}$$

where $Q_i = X_i F_i^\dagger$ and $R_i = H_i K_i^\dagger$. Since χ and χ^{-1} share the same pole, we have to impose $F_i^\dagger H_i = 0$ and consider

$$\Gamma_{ii} = -F_i^\dagger \psi'(\lambda_i) \psi^{-1}(\lambda_i) H_i + f_i c_i h_i. \tag{B.3}$$

There are three constraints imposed on $\psi(\lambda)$,

$$\left[\psi(\bar{\lambda}) \right]^\dagger = \psi^{-1}(\lambda), \qquad \left[\psi(\bar{\lambda}) \right]^* = \psi(\lambda), \qquad gK \psi(\lambda) K = \psi(1/\lambda), \tag{B.4}$$

where $g \equiv \psi(0)$ and K is an antisymmetric involution defined in (2.22). The first, unitarity constraint is solved by[14]

$$F_i = H_{\bar{i}}, \qquad \Gamma_{ii} = -\left(\Gamma_{\bar{u}} \right)^\dagger. \tag{B.5}$$

The second, orthogonality constraint by

$$F_i = \left(F_{\bar{i}} \right)^*, \qquad \Gamma_{ii} = \left(\Gamma_{\bar{u}} \right)^*. \tag{B.6}$$

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[14] $\lambda_{\bar{i}} = \bar{\lambda}_i$ and $\lambda_{\hat{i}} = 1/\lambda_i$.

The third, inversion constraint by

$$F_i = gK F_{\hat{i}}, \qquad \Gamma_{ii} = -\lambda_i^2 \, \Gamma_{\hat{i}\hat{i}} \, . \tag{B.7}$$

Two conditions (B.5) and (B.6) shows

$$F_i^{\mathrm{T}} F_i = f_i^{\mathrm{T}} f_i = 0, \qquad \Gamma_{ii}^{\mathrm{T}} = -\Gamma_{ii} \, . \tag{B.8}$$

It follows that $\Gamma_{ii} = 0$ when Γ_{ii} is a rank-one matrix.

If $|\lambda_1| = 1$, we have to solve $F_i^* = gK F_i$, which is equivalent to $f_i^* = Kf_i$. Since K is a real antisymmetric matrix, the conditions $f_i^* = K f_i$ means $0 = f_i^{\mathrm{T}} K f_i = -f_i^{\dagger} f_i$, namely $f_i = 0$. Therefore, the dressing matrix becomes trivial on the unit circle.[15]

References

[1] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nucl. Phys. B* **811** (2009) 66 [arXiv:0709.1260 [hep-th]].

[2] J. Bagger and N. Lambert, "Gauge Symmetry and Supersymmetry of Multiple M2-Branes," *Phys. Rev. D* **77** (2008) 065008 [arXiv:0711.0955 [hep-th]].

[3] J. Bagger and N. Lambert, "Comments on Multiple M2-branes," *JHEP* **0802** (2008) 105 [arXiv:0712.3738 [hep-th]].

[4] D. Gaiotto and E. Witten, "Janus Configurations, Chern-Simons Couplings, and the Theta-Angle in $\mathcal{N} = 4$ Super Yang-Mills Theory," arXiv:0804.2907 [hep-th].

[5] K. Hosomichi, K. M. Lee and S. Lee, "Mass-Deformed Bagger-Lambert Theory and its BPS Objects," *Phys. Rev. D* **78** (2008) 066015 [arXiv:0804.2519 [hep-th]].

[6] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **0810** (2008) 091 [arXiv:0806.1218 [hep-th]].

[7] M. Benna, I. Klebanov, T. Klose and M. Smedback, "Superconformal Chern-Simons Theories and AdS$_4$/CFT$_3$ Correspondence," *JHEP* **0809** (2008) 072 [arXiv:0806.1519 [hep-th]].

[8] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

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[15]Considering the fact that the overall normalization of f_i is unimportant in the dressing method, it might be possible to obtain a nontrivial profile in the limit $|\lambda_1| \to 1$, if one can take such limit in a sophisticated way.

[9] J. A. Minahan and K. Zarembo, "The Bethe ansatz for superconformal Chern-Simons," *JHEP* **0809** (2008) 040 [arXiv:0806.3951 [hep-th]].

[10] D. Bak and S. J. Rey, "Integrable Spin Chain in Superconformal Chern-Simons Theory," *JHEP* **0810** (2008) 053 [arXiv:0807.2063 [hep-th]].

[11] J. A. Minahan, W. Schulgin and K. Zarembo, "Two loop integrability for Chern-Simons theories with $\mathcal{N} = 6$ supersymmetry," *JHEP* **0903** (2009) 057 [arXiv:0901.1142 [hep-th]].

[12] B. I. Zwiebel, "Two-loop Integrability of Planar $\mathcal{N} = 6$ Superconformal Chern-Simons Theory," arXiv:0901.0411 [hep-th].

[13] G. Arutyunov and S. Frolov, "Superstrings on $AdS_4 \times \mathbb{CP}^3$ as a Coset Sigma-model," *JHEP* **0809** (2008) 129 [arXiv:0806.4940 [hep-th]].

[14] B. j. Stefanski, "Green-Schwarz action for Type IIA strings on $AdS_4 \times \mathbb{CP}^3$," *Nucl. Phys. B* **808** (2009) 80 [arXiv:0806.4948 [hep-th]].

[15] J. Gomis, D. Sorokin and L. Wulff, "The complete $AdS_4 \times \mathbb{CP}^3$ superspace for the type IIA superstring and D-branes," *JHEP* **0903** (2009) 015 [arXiv:0811.1566 [hep-th]].

[16] P. Fre and P. A. Grassi, "Pure Spinor Formalism for $Osp(\mathcal{N}|4)$ backgrounds," arXiv:0807.0044 [hep-th].

[17] G. Bonelli, P. A. Grassi and H. Safaai, "Exploring Pure Spinor String Theory on $AdS_4 \times \mathbb{CP}^3$," *JHEP* **0810** (2008) 085 [arXiv:0808.1051 [hep-th]].

[18] R. D'Auria, P. Frè, P. A. Grassi and M. Trigiante, "Superstrings on $AdS_4 \times \mathbb{CP}^3$ from Supergravity," *Phys. Rev. D* **79** (2009) 086001 [arXiv:0808.1282 [hep-th]].

[19] N. Berkovits, "Quantum consistency of the superstring in $AdS_5 \times S^5$ background," *JHEP* **0503** (2005) 041 [arXiv:hep-th/0411170].

[20] N. Gromov and P. Vieira, "The all loop AdS_4/CFT_3 Bethe ansatz," *JHEP* **0901** (2009) 016 [arXiv:0807.0777 [hep-th]].

[21] C. Ahn and R. I. Nepomechie, "$\mathcal{N} = 6$ super Chern-Simons theory S-matrix and all-loop Bethe ansatz equations," *JHEP* **0809** (2008) 010 [arXiv:0807.1924 [hep-th]].

[22] C. Ahn and R. I. Nepomechie, "An alternative S-matrix for $\mathcal{N} = 6$ Chern-Simons theory ?," *JHEP* **0903** (2009) 068 [arXiv:0810.1915 [hep-th]].

[23] C. Ahn and R. I. Nepomechie, "Two-loop test of the $\mathcal{N} = 6$ Chern-Simons theory S-matrix," *JHEP* **0903** (2009) 144 [arXiv:0901.3334 [hep-th]].

[24] T. Nishioka and T. Takayanagi, "On Type IIA Penrose Limit and $\mathcal{N} = 6$ Chern-Simons Theories," *JHEP* **0808** (2008) 001 [arXiv:0806.3391 [hep-th]].

[25] D. Gaiotto, S. Giombi and X. Yin, "Spin Chains in $\mathcal{N} = 6$ Superconformal Chern-Simons-Matter Theory," *JHEP* **0904** (2009) 066 [arXiv:0806.4589 [hep-th]].

[26] G. Grignani, T. Harmark and M. Orselli, "The $SU(2)$ x $SU(2)$ sector in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory," *Nucl. Phys. B* **810** (2009) 115 [arXiv:0806.4959 [hep-th]].

[27] D. Astolfi, V. G. M. Puletti, G. Grignani, T. Harmark and M. Orselli, "Finite-size corrections in the $SU(2)$ x $SU(2)$ sector of type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$," *Nucl. Phys. B* **810** (2009) 150 [arXiv:0807.1527 [hep-th]].

[28] P. Sundin, "The $\text{AdS}_4 \times \mathbb{CP}^3$ string and its Bethe equations in the near plane wave limit," *JHEP* **0902** (2009) 046 [arXiv:0811.2775 [hep-th]].

[29] T. McLoughlin and R. Roiban, "Spinning strings at one-loop in $\text{AdS}_4 \times \mathbb{P}^3$," *JHEP* **0812** (2008) 101 [arXiv:0807.3965 [hep-th]].

[30] L. F. Alday, G. Arutyunov and D. Bykov, "Semiclassical Quantization of Spinning Strings in $\text{AdS}_4 \times \mathbb{CP}^3$," *JHEP* **0811** (2008) 089 [arXiv:0807.4400 [hep-th]].

[31] C. Krishnan, "$\text{AdS}_4/\text{CFT}_3$ at One Loop," *JHEP* **0809** (2008) 092 [arXiv:0807.4561 [hep-th]].

[32] N. Gromov and V. Mikhaylov, "Comment on the Scaling Function in $\text{AdS}_4 \times \mathbb{CP}^3$," *JHEP* **0904** (2009) 083 [arXiv:0807.4897 [hep-th]].

[33] T. McLoughlin, R. Roiban and A. A. Tseytlin, "Quantum spinning strings in $\text{AdS}_4 \times \mathbb{CP}^3$: testing the Bethe Ansatz proposal," *JHEP* **0811** (2008) 069 [arXiv:0809.4038 [hep-th]].

[34] G. Grignani, T. Harmark, M. Orselli and G. W. Semenoff, "Finite size Giant Magnons in the string dual of $\mathcal{N} = 6$ superconformal Chern-Simons theory," *JHEP* **0812** (2008) 008 [arXiv:0807.0205 [hep-th]].

[35] B. H. Lee, K. L. Panigrahi and C. Park, "Spiky Strings on $\text{AdS}_4 \times \mathbb{CP}^3$," *JHEP* **0811** (2008) 066 [arXiv:0807.2559 [hep-th]].

[36] B. Chen and J. B. Wu, "Semi-classical strings in $\text{AdS}_4 \times \mathbb{CP}^3$," *JHEP* **0809** (2008) 096 [arXiv:0807.0802 [hep-th]].

[37] C. Ahn, P. Bozhilov and R. C. Rashkov, "Neumann-Rosochatius integrable system for strings on $\text{AdS}_4 \times \mathbb{CP}^3$," *JHEP* **0809** (2008) 017 [arXiv:0807.3134 [hep-th]].

[38] D. Bombardelli and D. Fioravanti, "Finite-Size Corrections of the $\mathbb{CP}^3$ Giant Magnons: the Lúscher terms," arXiv:0810.0704 [hep-th].

[39] T. Lukowski and O. O. Sax, "Finite size giant magnons in the $SU(2)$ x $SU(2)$ sector of AdS$_4 \times \mathbb{CP}^3$," *JHEP* **0812** (2008) 073 [arXiv:0810.1246 [hep-th]].

[40] C. Ahn and P. Bozhilov, "Finite-size Effect of the Dyonic Giant Magnons in $\mathcal{N} = 6$ super Chern-Simons Theory," arXiv:0810.2079 [hep-th].

[41] A. Agarwal, N. Beisert and T. McLoughlin, "Scattering in Mass-Deformed $\mathcal{N} \geq 4$ Chern-Simons Models," arXiv:0812.3367 [hep-th].

[42] O. Bergman and S. Hirano, "Anomalous radius shift in AdS$_4$/CFT$_3$," arXiv:0902.1743 [hep-th].

[43] N. Beisert, "The $su(2|2)$ dynamic S-matrix," *Adv. Theor. Math. Phys.* **12** (2008) 945 [arXiv:hep-th/0511082].

[44] N. Beisert, "The Analytic Bethe Ansatz for a Chain with Centrally Extended $su(2|2)$ Symmetry," *J. Stat. Mech.* **0701** (2007) P017 [arXiv:nlin/0610017].

[45] D. M. Hofman and J. M. Maldacena, "Giant magnons," *J. Phys. A* **39** (2006) 13095 [arXiv:hep-th/0604135].

[46] N. Gromov and P. Vieira, "The AdS$_4$/CFT$_3$ algebraic curve," *JHEP* **0902** (2009) 040 [arXiv:0807.0437 [hep-th]].

[47] I. Shenderovich, "Giant magnons in AdS$_4$/CFT$_3$: dispersion, quantization and finite–size corrections," arXiv:0807.2861 [hep-th].

[48] H. Y. Chen, N. Dorey and K. Okamura, "On the scattering of magnon boundstates," *JHEP* **0611** (2006) 035 [arXiv:hep-th/0608047].

[49] R. Roiban, "Magnon bound-state scattering in gauge and string theory," *JHEP* **0704** (2007) 048 [arXiv:hep-th/0608049].

[50] J. A. Minahan, "Zero modes for the giant magnon," *JHEP* **0702** (2007) 048 [arXiv:hep-th/0701005].

[51] G. Papathanasiou and M. Spradlin, "Semiclassical Quantization of the Giant Magnon," *JHEP* **0706** (2007) 032 [arXiv:0704.2389 [hep-th]].

[52] H. Y. Chen, N. Dorey and R. F. Lima Matos, "Quantum Scattering of Giant Magnons," *JHEP* **0709** (2007) 106 [arXiv:0707.0668 [hep-th]].

[53] N. Gromov and P. Vieira, "The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve," *Nucl. Phys. B* **789** (2008) 175 [arXiv:hep-th/0703191].

[54] B. Vicedo, "Semiclassical Quantisation of Finite-Gap Strings," *JHEP* **0806** (2008) 086 [arXiv:0803.1605 [hep-th]].

[55] V. E. Zakharov and A. V. Mikhailov, "Relativistically Invariant Two-Dimensional Models in Field Theory Integrable By the Inverse Problem Technique. (In Russian)," *Sov. Phys. JETP* **47** (1978) 1017 [Zh. Eksp. Teor. Fiz. **74** (1978) 1953].

[56] V.E. Zakharov and A.B. Shabat, "Integration of the nonlinear equations of mathematical physics by the method of the inverse scattering problem II" *Funct. Anal. Appl.* **13** (1979) 166. Also included in; C. Rebbi and G. Soliani (ed.), "Solitons and particles", *Singapore, World Scientific* (1984).

[57] V. E. Zakharov and A. V. Mikhailov, "On the Integrability of Classical Spinor Models in Two-Dimensional Space-Time," *Commun. Math. Phys.* **74** (1980) 21.

[58] J. P. Harnad, Y. Saint Aubin and S. Shnider, "Backlund Transformations for Nonlinear Sigma Models with Values in Riemannian Symmetric Spaces," *Commun. Math. Phys.* **92** (1984) 329.

[59] J. P. Harnad, Y. Saint Aubin and S. Shnider, "The Soliton Correlation Matrix and the Reduction Problem for Integrable Systems," *Commun. Math. Phys.* **93** (1984) 33.

[60] R. Sasaki, "Explicit Forms of Backlund Transformations for the Grassmannian and $\mathbb{CP}^{N-1}$ Sigma Models," *J. Math. Phys.* **26** (1985) 1786.

[61] J. P. Antoine and B. Piette, "Classical non-linear sigma models on Grassmann manifolds of compact or non-compact type," *J. Math. Phys.* **28** (1987) 2753.

[62] B. Piette, "Explicit Solutions of Grassmannian sigma Models," *J. Math. Phys.* **29** (1988) 2190.

[63] M. Spradlin and A. Volovich, "Dressing the giant magnon," *JHEP* **0610** (2006) 012 [arXiv:hep-th/0607009].

[64] C. Kalousios, M. Spradlin and A. Volovich, "Dressing the giant magnon. II," *JHEP* **0703** (2007) 020 [arXiv:hep-th/0611033].

[65] T. J. Hollowood and J. L. Miramontes, "Magnons, their Solitonic Avatars and the Pohlmeyer Reduction," *JHEP* **0904** (2009) 060 [arXiv:0902.2405 [hep-th]].

[66] C. Kalousios, M. Spradlin and A. Volovich, "Dyonic Giant Magnons on $\mathbb{CP}^3$," arXiv:0902.3179 [hep-th].

[67] M. C. Abbott and I. Aniceto, "Giant Magnons in $AdS_4 \times \mathbb{CP}^3$: Embeddings, Charges and a Hamiltonian," arXiv:0811.2423 [hep-th].

[68] H. Eichenherr and M. Forger, "More About Nonlinear Sigma Models on Symmetric Spaces," *Nucl. Phys. B* **164** (1980) 528 [Erratum-ibid. B **282** (1987) 745].

[69] H. Y. Chen, N. Dorey and K. Okamura, "Dyonic giant magnons," *JHEP* **0609** (2006) 024 [arXiv:hep-th/0605155].

[70] K. Okamura and R. Suzuki, "A perspective on classical strings from complex sine-Gordon solitons," *Phys. Rev. D* **75** (2007) 046001 [arXiv:hep-th/0609026].

[71] K. Pohlmeyer, "Integrable Hamiltonian Systems and Interactions Through Quadratic Constraints," *Commun. Math. Phys.* **46** (1976) 207.

[72] F. Lund and T. Regge, "Unified Approach to Strings and Vortices with Soliton Solutions," *Phys. Rev. D* **14** (1976) 1524.

[73] N. Beisert, R. Hernandez and E. Lopez, "A crossing-symmetric phase for $AdS_5 \times S^5$ strings," *JHEP* **0611** (2006) 070 [arXiv:hep-th/0609044].

[74] J. M. Maldacena and I. Swanson, "Connecting giant magnons to the pp-wave: An interpolating limit of $AdS_5 \times S^5$," *Phys. Rev. D* **76** (2007) 026002 [arXiv:hep-th/0612079].

[75] N. Dorey, D. M. Hofman and J. M. Maldacena, "On the singularities of the magnon S-matrix," *Phys. Rev. D* **76** (2007) 025011 [arXiv:hep-th/0703104].

[76] N. Dorey and K. Okamura, "Singularities of the Magnon Boundstate S-Matrix," *JHEP* **0803** (2008) 037 [arXiv:0712.4068 [hep-th]].

[77] P. Bozhilov, "Probe branes dynamics: Exact solutions in general backgrounds," *Nucl. Phys. B* **656** (2003) 199 [arXiv:hep-th/0211181].

[78] P. Bozhilov, "Neumann and Neumann-Rosochatius integrable systems from membranes on $AdS_4 \times S^7$," *JHEP* **0708** (2007) 073 [arXiv:0704.3082 [hep-th]].

[79] P. Bozhilov and R. C. Rashkov, "On the multi-spin magnon and spike solutions from membranes," *Nucl. Phys. B* **794** (2008) 429 [arXiv:0708.0325 [hep-th]].

[80] C. Ahn and P. Bozhilov, "M2-brane Perspective on $\mathcal{N} = 6$ Super Chern-Simons Theory at Level k," *JHEP* **0812** (2008) 049 [arXiv:0810.2171 [hep-th]].

[81] R. D'Auria, T. Regge and S. Sciuto, "A General Scheme for Bidimensional Models with Associate Linear Set," *Phys. Lett. B* **89** (1980) 363.

[82] R. D'Auria, T. Regge and S. Sciuto, "Group Theoretical Construction of Two-Dimensional Models with Infinite Set of Conservation Laws," *Nucl. Phys. B* **171** (1980) 167.

[83] H. Eichenherr and J. Honerkamp, "Reduction of the $\mathbb{CP}^N$ Nonlinear Sigma Model," *J. Math. Phys.* **22** (1981) 374.

[84] I. Bakas, Q. H. Park and H. J. Shin, "Lagrangian Formulation of Symmetric Space sine-Gordon Models," *Phys. Lett. B* **372** (1996) 45 [arXiv:hep-th/9512030].

[85] C. R. Fernandez-Pousa, M. V. Gallas, T. J. Hollowood and J. L. Miramontes, "The symmetric space and homogeneous sine-Gordon theories," *Nucl. Phys. B* **484** (1997) 609 [arXiv:hep-th/9606032].

[86] J. L. Miramontes, "Pohlmeyer reduction revisited," *JHEP* **0810** (2008) 087 [arXiv:0808.3365 [hep-th]].

[87] R. C. Rashkov, "A note on the reduction of the $\text{AdS}_4 \times \mathbb{CP}^3$ string sigma model," *Phys. Rev. D* **78** (2008) 106012 [arXiv:0808.3057 [hep-th]].